J. Fluid Mech. (1991), *vol.* 223, *pp.* 521–536
Printed in Great Britain

Passive scalar transport in β-plane turbulence

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(Received 16 October 1989 and in revised form 31 July 1990)

Evaluation of spectral closure theory and direct numerical simulation are used to examine the eddy transport of a passive scalar in barotropic β-plane flow. When a large-scale gradient of scalar concentration is imposed, the implied scale separation between fixed background gradient and eddies supports the concept of 'eddy diffusion'. The results can be cast in terms of an eddy diffusion tensor $\boldsymbol{K}$, whose behaviour as a function of mean vorticity gradient β is examined. Earlier theoretical work by Holloway & Kristmannsson (1984) is extended to include cases where strong vorticity–scalar correlations are observed, and corrected in order to restore random Galilean invariance.

The anisotropy of eddy energy and the direct influence of Rossby wave propagation contribute to the overall anisotropy of $\boldsymbol{K}$. The resulting suppression of meridional diffusivity K_{yy}, and enhancement of zonal diffusivity K_{xx}, with increased β is examined. The variation in simulation K_{yy} is closely reproduced in the closure equations. However, the increased K_{xx} is the result of zonal jets whose persistence is not accounted for in the statistical theory.

1. Introduction

A major obstacle to the study of the large-scale distribution of various substances in the atmosphere and oceans is the lack of understanding of turbulent transport. Simple specification of eddy diffusivities in analogy with thermodynamic properties of the fluid is too crude to be of much value. Actual turbulent diffusion is a complex dynamic process influenced by all aspects of the physical setting. For example, on the sphere, Rossby wave propagation produces anisotropy in the velocity field, resulting in anisotropic diffusion. A quantitatively skilful understanding of turbulent mixing in the presence of Rossby waves is a necessary first step towards predictive capability with respect to large-scale transport of passive scalars in geophysical fluids. Applications of current interest include the dispersion of pollutants, the oceanic heat, salt and CO_2 fluxes, the horizontal transport of stratospheric ozone, etc.

The spectral transfer of passive scalar variance in two-dimensional turbulence has been studied in the absence of β by Lesieur, Sommeria & Holloway (1981) and Lesieur & Herring (1985). Although the vorticity and scalar variance are governed by similar advective dynamics, their statistical evolutions can be quite different owing to the lack of a scalar counterpart to the relationship between the vorticity and velocity fields as seen in the numerical experiments of Babiano *et al.* (1987). In their simulations both vorticity and scalar variance were injected by fixing the amplitude of one of the large-scale modes. At smaller scales the enstrophy spectrum was considerably steeper than that of the scalar variance while the vorticity field was

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dominated by coherent vortices (cf. Fornberg 1977; Basdevant *et al.* 1981; McWilliams 1984) whereas the scalar field was not. Since most of the simulations to be discussed in the present study included β, and all were forced with an approximate white-noise time dependence, coherent vorticity structures did not form (Holloway 1984; McWilliams 1984; Herring & McWilliams 1985), and thus do not adversely affect the results.

Holloway & Kristmannsson (1984, hereinafter referred to as HK) considered a barotropic β-plane flow advecting a passive scalar whose concentration consisted of two components: a fixed constant background gradient and an active eddy field. The down-gradient transport is effected by the eddies, which are not permitted to deplete the gradient component of the field. Since the persistence of the gradient acts to inject variance into the large scales of the eddy component, and the scalar variance transfer is predominantly down-scale, statistical stationarity can be achieved in the presence of small-scale diffusion.

The scalar flux can be used to define the diffusivity tensor, $\boldsymbol{K}$. If $\phi(\boldsymbol{r},t)$ represents the scalar eddy concentration, and $\boldsymbol{G}$ the fixed gradient, then

$$\langle \boldsymbol{u}\phi\rangle = -\boldsymbol{K}\cdot\boldsymbol{G}, \tag{1}$$

where $\boldsymbol{u} = u\hat{\imath}+v\hat{\jmath}$ is the eddy velocity field and $\langle - \rangle$ denotes the ensemble average. HK used a relatively simple closure hypothesis (the Markovian Random Coupling Model (Frisch, Lesieur & Brissaud 1974)) as the basis for their statistical theory. Agreement with some aspects of numerical simulation led Holloway (1986) and Keffer & Holloway (1988) to propose a highly simplified abridgement of the theory, expressed as

$$K_{yy} = \frac{C\psi_{\mathrm{rms}}}{(B\hat{\beta})^2+1}, \tag{2}$$

for the meridional diffusivity K_{yy}, where B and C are constants, $\hat{\beta} = \beta u_{\mathrm{rms}}/\zeta^2_{\mathrm{rms}}$ and $(\psi, u, \zeta)_{\mathrm{rms}}$ are the root-mean-square stream function, speed and vorticity respectively. Equation (2) was compared with direct numerical simulation, giving reasonable agreement when $C \approx 0.4$ and $B \approx 3$. The appeal of (2) is the simplicity of its implementation in numerical models as the basis for a parameterization of K_{yy}.

The purpose of the present paper is to compare the full theory with direct numerical simulation and to evaluate its potential for quantitative predictive skill with respect to $\boldsymbol{K}$. For this purpose the Random Coupling Model was replaced with the Test Field Model (TFM) (Kraichnan 1971) which, although more complicated, has a history of favourable comparison with direct numerical simulation of two-dimensional flow (Herring *et al.* 1974; Herring & McWilliams 1985). The TFM, unlike the direct interaction approximation (DIA) (see e.g. Leslie 1973) is invariant to random Galilean transformation in that a spatially uniform advection of the velocity field, varying randomly from realization to realization, does not influence the statistics of the variance transfer. This deficiency of the DIA makes it unsuitable for comparison with simulations containing a wide range of scales.

The ability of the theory to duplicate simulation statistics in the presence of β depends on a number of unresolved issues. For example, Legras (1980) and Carnevale & Martin (1982) have derived turbulent Rossby wave frequency shifts which are not accounted for in this study. Also, Shepherd (1987), through the use of finite-amplitude stability constraints applied to the inviscid system, has shown that β-plane flow cannot be ergodic for sufficiently large β, implying that the statistical mechanical hypothesis of equal *a priori* probability on phase space surfaces of constant energy and enstrophy may not be valid. Ergodicity forms an integral part of the closure apparatus where the philosophy is to assume 'maximum randomness' apart from energy and enstrophy conservation by the transfer terms. The consequence of these stability constraints may be stable zonal jets as seen in the simulations of Panetta & Held (1988). Although isolated vorticity structures are not observed in the presence of significant β, another form of phase coherence was noted in their simulations. Using a simplified baroclinic β-plane model containing a wide range of scales, they observed remarkably persistent zonal jets which displayed little tendency to wander in the meridional direction and often remained intact for hundreds of large-scale turnover times. If this behaviour is even partially shared by the barotropic model, it is not expected to be reproduced in the closure theory which is unable to account for the necessary phase coherence. This would also apply to the scalar advection by the jets.

The model details are discussed in §2 and the theory–simulation comparisons are presented in §3. Both β and the direction of the spatially uniform scalar gradient are varied. Although the meridional diffusivity is treated fairly well by the theory, the enhancement of the zonal diffusivity with β is not a feature of the closure model. This is due to large zonal-mode phase persistence times in the simulations which significantly exceed estimates based on eddy turnover times. The resulting persistent zonal jets act to advect scalar material more efficiently in the simulations than in the statistical theory. The conclusions form §4.

2. Model equations and closure theory

Following HK we write down the equations describing a passive scalar field coevolving with the vorticity in a barotropic β-plane setting. If the total passive tracer concentration is given by $\boldsymbol{\Phi}(\boldsymbol{r},t) = \phi_0+\boldsymbol{G}\cdot\boldsymbol{r}+\phi(\boldsymbol{r},t)$, where $\boldsymbol{G}$ is a fixed background gradient, then the vorticity $\zeta(\boldsymbol{r},t)$ and scalar eddy component $\phi(\boldsymbol{r},t)$ are governed by

$$\frac{\partial}{\partial t}\zeta+J(\psi,\zeta+\beta y) = \mathrm{D}_1\zeta+\xi, \tag{3}$$

$$\frac{\partial}{\partial t}\phi+J(\psi,\phi+\boldsymbol{G}\cdot\boldsymbol{r}) = \mathrm{D}_2\phi, \tag{4}$$

where $\zeta = \nabla^2\psi$, D_i are linear operators acting to dissipate fluctuation variance and ξ represents external vorticity sources. After imposing periodic boundary conditions in both dimensions, an exact Fourier representation can be obtained:

$$\left(\frac{\partial}{\partial t}+\mathrm{i}\omega_k+\nu_k\right)\zeta_k = f_k+\sum_{p+q=k} A_{kp}\zeta_p\zeta_q, \tag{5}$$

$$\left(\frac{\partial}{\partial t}+\kappa_k\right)\phi_k = \mathrm{i}G_k\zeta_k+\sum_{p+q=k} A_{kp}\zeta_p\phi_q, \tag{6}$$

where ν_k, κ_k are algebraic functions of $k = |\boldsymbol{k}|$ representing explicit vorticity and scalar dissipation due to operators D_i, $A_k = \hat{\boldsymbol{z}}\cdot(\boldsymbol{k}\times\boldsymbol{p})/p^2$, $G_k = \hat{\boldsymbol{z}}\cdot(\boldsymbol{G}\times\boldsymbol{k})/k^2$ and f_k is the transform of the external torque field ξ. The linear Rossby frequency is $\omega_k = -\beta k_x/k^2$.

For many purposes (5) and (6) provide more information than is desired or needed. When the location and intensities of individual eddies are not in question but rather some measure of their long-term effect, as in the case of subgrid-scale parameterization, a statistical approach is adopted and equations for the second moments are formulated. If these are expressed as $Z_k = \langle \zeta_k \zeta_{-k} \rangle$, $Q_k = \langle \phi_k \phi_{-k} \rangle$ and $\Gamma_k = \langle \zeta_k \phi_{-k} \rangle$, then

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial t} + 2\nu_k\right) Z_k &= F_k + 2 \sum_{p+q=k} A_{kp} \operatorname{Re} \langle \zeta_{-k} \zeta_p \zeta_q \rangle, \\
\left(\frac{\partial}{\partial t} + 2\kappa_k\right) Q_k &= -2G_k \operatorname{Im} \Gamma_k + 2 \sum_{p+q=k} A_{kp} \operatorname{Re} \langle \phi_{-k} \zeta_p \phi_q \rangle, \\
\left(\frac{\partial}{\partial t} + \mathrm{i}\omega_k + \nu_k + \kappa_k\right) \Gamma_k &= -\mathrm{i} G_k Z_k + \langle f_k \phi_{-k} \rangle + \sum_{p+q=k} (A_{kp} \langle \zeta_k \zeta_{-p} \phi_{-q} \rangle \\
&\quad + A_{kp} \langle \phi_{-k} \zeta_p \zeta_q \rangle).
\end{aligned}\right\} \tag{7}$$

Deriving the equations for successively higher-order moments would lead to an unclosed set (see e.g. Leslie 1973; Orszag 1977; Lesieur 1987). If the moments are written as products of lower-order moments plus a residual or cumulant, closure can be effected by assuming that fourth cumulants act to relax triple moments. Consideration is restricted to quasi-stationary statistics, i.e. that triple moments evolve on timescales short compared with those of second moments. If the fourth-cumulant terms in the triple-moment equations are replaced by damping terms characterized by rates $\mu_{kpq}^{(i)}$, then the stationary triple moments are written

$$\begin{aligned}
\langle \zeta_{-k} \zeta_p \zeta_q \rangle &= \theta_{-kpq}^{(1)} \{ (A_{kp} + A_{kq}) Z_p Z_q + (A_{pk} + A_{p-q}) Z_k Z_q + (A_{qk} + A_{q-p}) Z_k Z_p \}, \\
\langle \phi_{-k} \zeta_p \phi_q \rangle &= \theta_{-kpq}^{(2)} \{ A_{kp} Z_p Q_q + A_{q-p} Z_p Q_k + (A_{pk} + A_{p-q}) \Gamma_k \Gamma_{-q} + A_{qk} \Gamma_k \Gamma_p + A_{kq} \Gamma_p \Gamma_{-q} \}, \\
\langle \phi_{-k} \zeta_p \zeta_q \rangle &= \theta_{-kpq}^{(3)} \{ A_{kp} Z_p \Gamma_q + A_{kq} \Gamma_p Z_q + (A_{pk} + A_{p-q}) \Gamma_k Z_q + (A_{qk} + A_{q-p}) Z_p \Gamma_k \},
\end{aligned}$$

where, if we ignore frequency shifts of the kind proposed by Legras (1980) and Carnevale & Martin (1982),

$$\left.\begin{aligned}
\theta_{-kpq}^{(i)} &= \frac{\mu_{kpq}^{(i)} + \nu_{kpq}^{(i)}}{(\mu_{kpq}^{(i)} + \nu_{kpq}^{(i)})^2 + \omega_{-kpq}^{(i)^2}}, \\
\mu_{kpq}^{(i)} &= \mu_k^{(i)} + \mu_p^{(i)} + \mu_q^{(i)}, \\
\nu_{kpq}^{(1)} = \nu_k + \nu_p + \nu_q, \quad \nu_{kpq}^{(2)} &= \kappa_k + \nu_p + \kappa_q, \quad \nu_{kpq}^{(3)} = \kappa_k + \nu_p + \nu_q, \\
\omega_{-kpq}^{(1)} = \omega_{-k} + \omega_p + \omega_q, \quad \omega_{-kpq}^{(2)} &= \omega_p, \quad \omega_{-kpq}^{(3)} = \omega_p + \omega_q,
\end{aligned}\right\} \tag{8}$$

and linear gradient terms have been neglected.

The timescale for triple-moment relaxation $\mu_{-kpq}^{(1)}$ is obtained in the TFM from the rate at which advection induces exchange between the solenoidal and compressive parts of a compressible test field advected by the turbulent velocity field (Kraichnan 1971). The particular case of two-dimensional turbulence with Rossby waves has been examined by Holloway & Hendershott (1977) and Legras (1980). Following the former we take

$$\mu_k^{(1)} = g^{(1)^2} \sum_{p+q=k} \theta_{-kpq}^{(1)} \frac{|\boldsymbol{k} \times \boldsymbol{p}|^4}{k^2 p^4 q^2} Z_p, \tag{9}$$

where $g^{(1)}$, which represents the efficiency of pressure scrambling in decorrelating triple moments, is an adjustable parameter.

It remains to specify the $\mu_k^{(2,3)}$ and hence the $\theta_{-kpq}^{(2,3)}$ in such a way as to ensure that the closure respects the realizability conditions that $Z_k \geqslant 0$, $Q_k \geqslant 0$ and $\Gamma_k \Gamma_{-k} \leqslant Z_k$

Q_k. A number of choices have been examined in previous work, most of which was restricted to the case $\langle \zeta\phi \rangle = 0$ and $\beta = 0$. For example, following Kraichnan (1971), two studies (Newman & Herring 1979; Herring *et al.* 1982) have specified two separate ways to calculate scalar transfer timescales based on the identification of the compressive part of the test field with the gradient field of the scalar. Other studies using eddy-damped quasi-normal Markovian closures (EDQNM) have also introduced multiplicative constants used in calculating the $\theta^{(i)}_{-kpq}$ (Larchevêque & Lesieur 1981; Lesieur & Herring 1985). It is clear that, within the limits imposed by the realizability constraints, there are a number of possibilities which require specification of further adjustable parameters. Given that this is the weak point of closure, we choose instead the simplest form $\theta^{(i)}_{-kpq} = \theta^{(1)}_{-kpq}$ with $g^{(i)} = 1$, for all triple moments. We have also performed numerical evaluations employing a number of representations involving different $\theta^{(i)}_{-kpq}$ as suggested by a reviewer. We find that the results are not very sensitive to these choices, at least within the parameter range that we have explored. In addition, equal $\theta^{(i)}_{-kpq}$ can be argued from consideration of an extreme case. If, for example, $\nu_k = \kappa_k, f_k = 0$ and the gradient is set to $\boldsymbol{G} = (0, \beta)$ with the initial fields satisfying $\zeta = \phi$, then (5) and (6) imply that the vorticity and scalar fields remain equal. Setting $Z_k = Q_k = \Gamma_k$ in (7) yields identical correlation, vorticity and scalar statistics only if the $\theta^{(i)}_{kpq}$ are all equal. Since the TFM provides a means of determining $\theta^{(1)}_{kpq}$, this has been adopted and superscripts will henceforth be dropped. The closed set then becomes,

$$\left(\frac{\partial}{\partial t} + 2\nu_k + 2\eta_k\right) Z_k = F_k + \Lambda_k, \tag{10}$$

$$\left(\frac{\partial}{\partial t} + 2\kappa_k + 2\gamma_k\right) Q_k = -2G_k \operatorname{Im} \Gamma_k + \Phi_k + 2\operatorname{Re}\{\sigma_{-k}\Gamma_k\}, \tag{11}$$

$$\left(\frac{\partial}{\partial t} + \mathrm{i}\omega_k + \nu_k + \kappa_k + \eta_k + \gamma_k\right) \Gamma_k = -\mathrm{i}G_k Z_k + \Pi_k + \sigma_k Z_k, \tag{12}$$

where
$$\eta_k = \sum_{p+q=k} \theta_{-kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left(\frac{1}{p^2} - \frac{1}{q^2}\right)\left(\frac{1}{p^2} - \frac{1}{k^2}\right) Z_p,$$

$$\Lambda_k = \sum_{p+q=k} \theta_{-kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left(\frac{1}{p^2} - \frac{1}{q^2}\right)^2 Z_p Z_q,$$

$$\gamma_k = \sum_{p+q=k} \theta_{-kpq} |\boldsymbol{k}+\boldsymbol{p}|^2 \frac{1}{p^4} Z_p,$$

$$\Phi_k = 2\operatorname{Re} \sum_{p+q=k} \theta_{-kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left(\frac{1}{p^4} Z_p Q_q - \frac{1}{p^2 q^2} \Gamma_p \Gamma_{-q}\right),$$

$$\sigma_k = \sum_{p+q=k} \theta_{-kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \left(\frac{1}{q^2}\left(\frac{1}{p^2} - \frac{1}{k^2}\right)\Gamma_p + \frac{1}{p^2 k^2}\Gamma_{-p}\right),$$

$$\Pi_k = 2\sum_{p+q=k} \theta_{-kpq} |\boldsymbol{k}\times\boldsymbol{p}|^2 \frac{1}{p^2}\left(\frac{1}{p^2} - \frac{1}{q^2}\right) Z_p \Gamma_q,$$

and the forcing term $\langle f_k \phi_{-k} \rangle$ in the correlation (Γ_k) equation has been omitted corresponding to the case of white-noise forcing considered here.

Equations (8), (9) (dropping superscripts), (10), (11) and (12) now form a closed set for the second moments. In order for the closure equations to mimic the nonlinear transfer of (3) and (4) as well as their inviscid equilibria (setting aside concerns regarding ergodicity in the presence of large β (Shepherd 1987)), it is necessary that the quadratic invariants of the nonlinear terms (energy, enstrophy, total eddy scalar content and scalar–vorticity correlation) survive the closure. This is ensured by the symmetry of θ_{-kpq} with respect to wavenumber. Equations (10), (11) and (12) also respect the realizability constraints mentioned above. Although the equations of HK, obtained by setting σ_k, Π_k and the second term in the summand for Φ_k to zero, do not satisfy Galilean invariance, the inclusion of the neglected terms restores it.

3. Numerical experiments

Although $\boldsymbol{K}$ is independent of $\boldsymbol{G}$ and can be obtained from experiments with a scalar gradient $\boldsymbol{G} = (G_1, G_2)$, where $G_i \neq 0$, the scalar variances will be affected if the theory is inaccurate in its prediction of either K_{xx} or K_{yy}. For this reason, the numerical experiments are organized into two groups. The first of these uses $\boldsymbol{G} = (0, 1)$ in order to focus on the meridional diffusion in a case where the ensemble-average zonal diffusion is zero, thus nullifying any effects due to possible persistent zonal jets. In the second group the gradient is set to $\boldsymbol{G} = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$ yielding equal zonal and meridional fluxes in the absence of β. This choice was motivated by the results of the first group of experiments, which show the theoretical K_{yy} to be accurate, and the fact that $G_1 = G_2$ provides a more thorough test of the closure theory since the vorticity–scalar correlation vanishes when $G_2 = 0$.

Although each simulation was forced from a state of rest, a statistically stationary state was eventually achieved. During this period the large-scale eddy turnover time, defined as $\tau = 2\pi/\zeta_{\mathrm{rms}}$, fell in the range 2.8 ± 0.1 for all experiments described in this study. We therefore adopt this value as our unit of time in the sequence. The initial spin-up period, approximately 20τ, was followed by the accumulation of statistics for another 90τ. The white-noise vorticity forcing was applied isotropically over the range $2 \leqslant k \leqslant 4$ and the vorticity and scalar dissipation functions were respectively $\nu_k = \nu_0 + \nu_4 k^4$ and $\kappa_k = \kappa_4 k^4$ with $\nu_0 = 0.05$ (yielding a spin-down time of 7τ) and $\nu_4 = \kappa_4 = 5 \times 10^{-6}$. Simulations using high-order and Laplacian viscosities have been compared by several authors (e.g. Basdevant & Sadourny 1983) with the conclusion that the use of the former does not significantly influence the statistics of the larger scales. Owing to the computationally demanding task of evaluating anisotropic closure equations, this study has been limited to modest resolution. High-order dissipation operators allow us to extend the effective Reynolds number, defined as a function of the ratio of the inner to outer scales. The Fourier representations were isotropically truncated at $k = k_{\mathrm{T}} = 30$ while dealiased convolutions were evaluated in real space using 64×64 collocation points. Other aspects of the numerical method are discussed in the Appendix.

3.1. *The case* $\boldsymbol{G} = (0, 1)$

A set of simulation–theory comparison experiments were run with the above parameters and β-values of 0, 1, 2, 3, 4 and 5, resulting in geophysically relevant values in the range $0 \leqslant \hat{\beta} \leqslant 0.9$. Figure 1 shows the convergence of average enstrophy, scalar variance and the components of the scalar flux

$$\langle \boldsymbol{u}\phi \rangle = \sum_{\boldsymbol{k}} \frac{\boldsymbol{k} \times \hat{\boldsymbol{z}}}{k^2} \mathrm{Im}\,\{\Gamma_{\boldsymbol{k}}\}$$

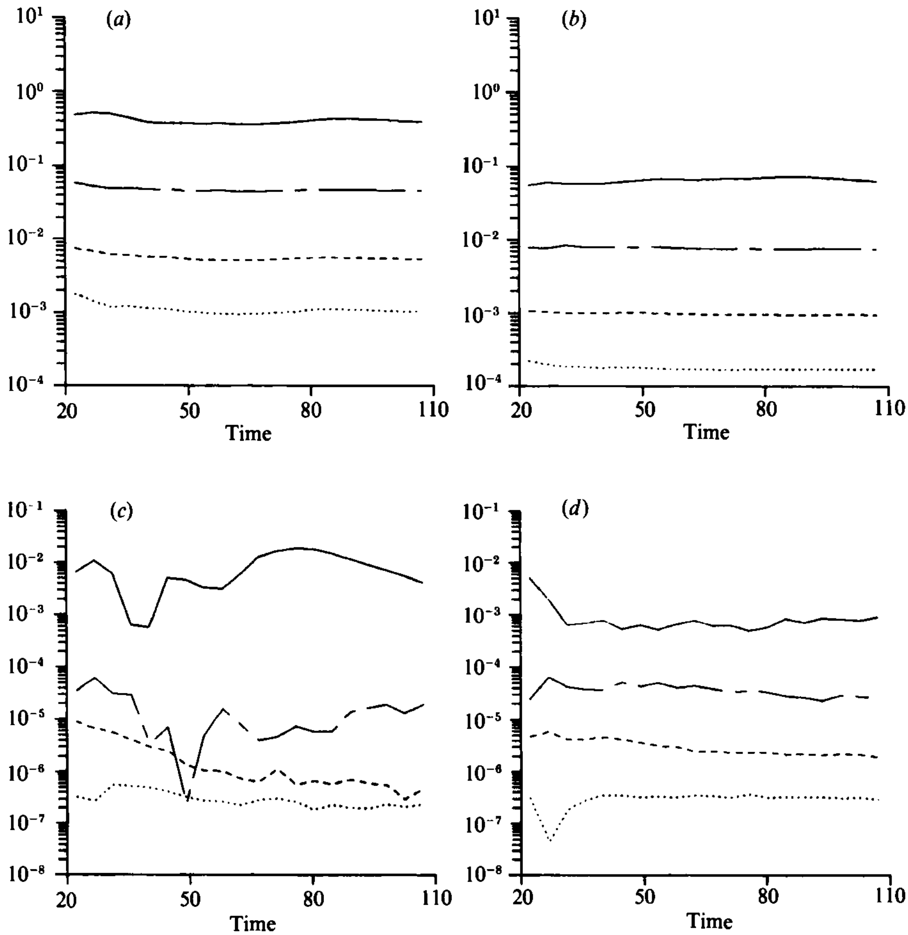

FIGURE 1. Time series of the cumulative average variance in the spectrum at wave-bands $k = 1$ (——), $k = 10$ (— — —), $k = 20$ (-----) and $k = 30$ (·····) for (*a*) the enstrophy, (*b*) the scalar variance, (*c*) the zonal flux and (*d*) the meridional flux ($\boldsymbol{G} = (0, 1)$).

in the wavebands centred on $k = 1$, 10, 20 and 30 for the case $\beta = 5$ ($\hat{\beta} = 0.9$) as a typical example. The convergence of average quantities is quite good, with the exception of the zonal scalar flux $\langle u\phi \rangle$. In the next section we shall present evidence that this is due to the existence of persistent zonal jets.

In agreement with previous studies including β or random agitation or both (Holloway 1984; McWilliams 1984; Herring & McWilliams 1985), the vorticity field was not visibly intermittent. This can be quantified by the vorticity kurtosis, $\mathscr{K}(\zeta) = \langle \zeta^4 \rangle / \langle \zeta^2 \rangle^2$, which was measured at the end of each run ($t > 100\tau$) and took the near-Gaussian values of 3.4 and 2.6 for the cases $\beta = 0$ and 5, respectively. Although these experiments did not show large vorticity kurtoses, it has been argued by Herring & McWilliams (1985) that the lack of intermittency in the stream function may imply that large-scale quantities such as eddy transports are described well by the closure theory even in the presence of isolated vortices. Another aspect of the appearance of the fields was the increasing anisotropy with β. A measure of this is $\mathscr{A} = (u^2_{\mathrm{rms}} - v^2_{\mathrm{rms}})/(u^2_{\mathrm{rms}} + v^2_{\mathrm{rms}})$, which was found to be 0.007 for $\beta = 0$ and 0.181 for $\beta = 5$.

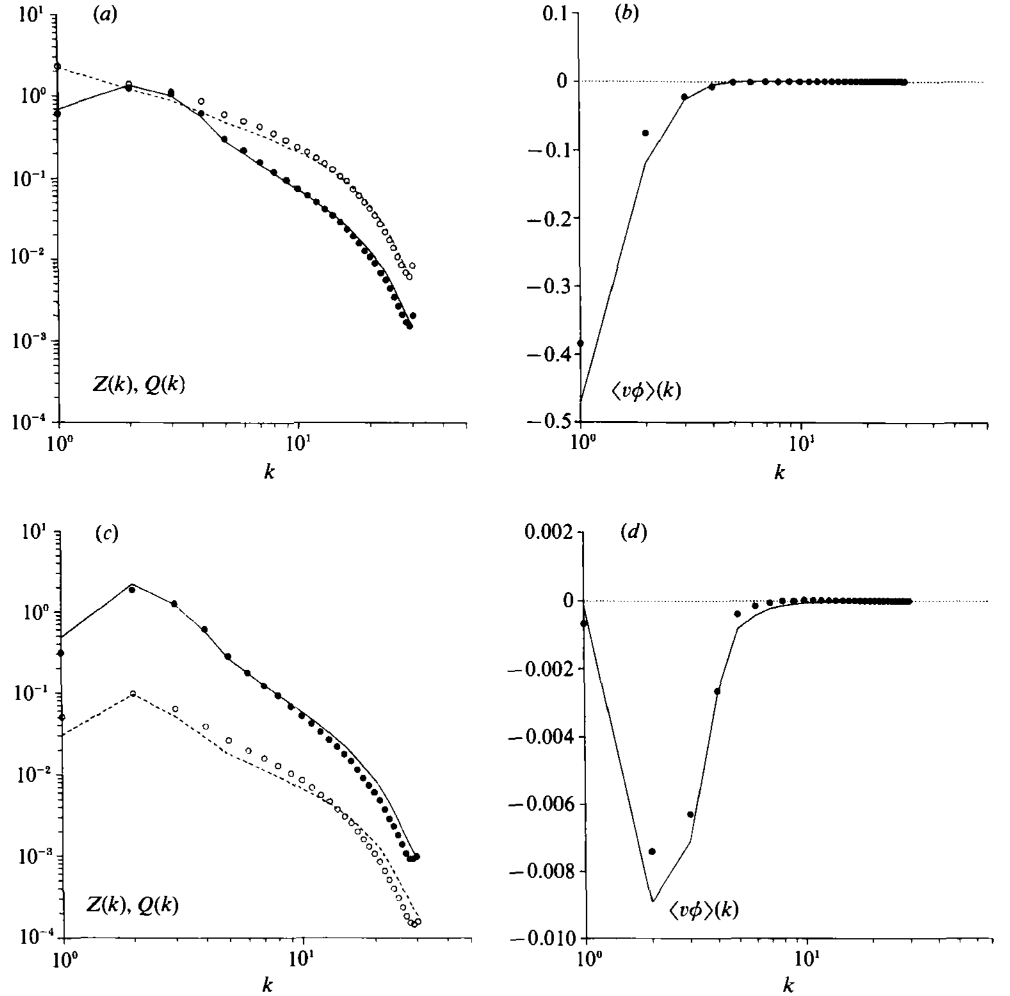

FIGURE 2. The vorticity (——, ●) and scalar (-----, ○) variance spectra as well as the meridional flux spectrum for (*a*, *b*) $\beta = 0$ and (*c*, *d*) $\beta = 5$ ($\boldsymbol{G} = (0, 1)$), where the simulation and closure theory data are represented by circles and lines, respectively.

Figure 2 shows a comparison between the closure theory and the simulation of the vorticity and scalar variance spectra as well as the spectrum of the meridional scalar flux $\langle v\phi\rangle$ for $\beta = 0$ and 5 ($\hat{\beta} = 0$ and 0.9). The quality of the agreement demonstrates that the statistical theory is able to describe the very large variation in the scalar variance and down-gradient flux with β. The theory of HK, which effectively amounts to neglecting products involving the correlation Γ_k in the transfer terms, would not have been as successful when $\beta \neq 0$. For example, in the case of the $\beta = 5$ simulation $|\Gamma(k)| \geqslant Q(k)$ for all k and $\Sigma_k Z_k = 5.0$, $\Sigma_k \Gamma_k = 0.90$ and $\Sigma_k Q_k = 0.39$. Some systematic discrepancies are noted in the present theory however, such as the overestimation of the enstrophy spectrum by the TFM in the small scales. This has been ascribed to the influence of dissipation-range intermittency by Herring & McWilliams (1985) but may be complicated here by the unaccounted-for effect of the Robert filter (see the Appendix). We have also observed that the closure increasingly overestimates the energy in the largest scales as β is increased, and attribute it to the

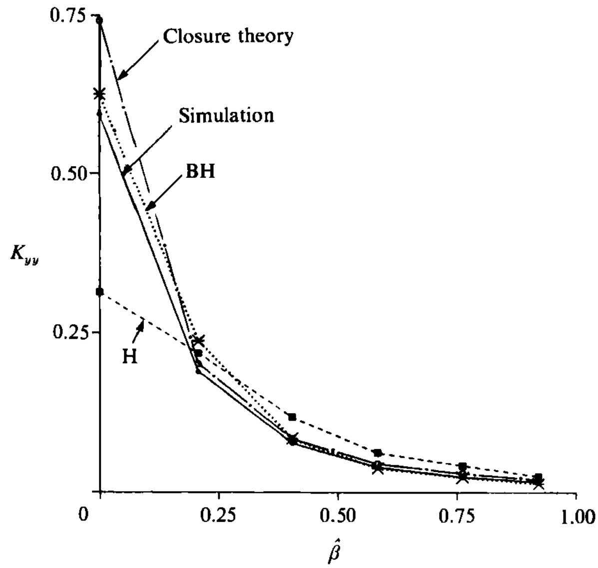

FIGURE 3. K_{yy} *vs.* $\hat{\beta}$ for simulation, closure theory and the expression (2) with $A = 0.4$ and $B = 3$ (labelled H) and $A = 0.8$ and $B = 6$ (labelled BH) ($\boldsymbol{G} = (0, 1)$).

stabilizing effect of β on zonal currents. Other closure–simulation discrepancies are observed in the scalar statistics: the magnitude of the large-scale down-gradient flux is overestimated and the scalar variance is underestimated at intermediate scales and overestimated at small scales. Not surprisingly, a more careful treatment of the $\theta^{(i)}_{-kpq}$ using (8) with $\mu^{(i)}_k = \mu^{(1)}_k$ as given by (9) also produced spectra which were quite close to the simulation values, while showing similar systematic differences.

Figure 3 displays the meridional diffusivity, K_{yy} as a function of $\hat{\beta}$ for all of the experiments as well as for the closure theory. Once again, the statistical theory was able to account fairly accurately for an order of magnitude variation due to the influence of β. The results obtained from two versions of the simplified expression (2) are also displayed. The curve labelled 'H', which was obtained using the Holloway (1986) values $C = 0.4$ and $B = 3$, seriously underestimates K_{yy} at $\beta = 0$ and falls off too slowly with β at larger values. Although many of the considerations underlying (2) are complicated by the addition of the new terms included in the present study, it was found that this expression, with $A = 0.8$ and $B = 6$ provided a good fit to the simulation data (see the curve labelled 'BH').

3.2. *The case* $\boldsymbol{G} = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$

An experiment with equal zonal and meridional gradient components was run in the same manner as above in order to investigate the zonal diffusivity. The spectral comparison (as in figure 2) of the $\beta = 5$ case is shown in figure 4, where it can be seen that the closure underestimated the magnitude of the zonal flux and the scalar variance by a wide margin. The closure evaluation using different $\theta^{(i)}_{-kpq}$ as prescribed by (8) showed the same discrepancy but to a lesser extent. Still, the large-scale scalar variance and zonal flux were systematically underestimated by factors of about 10 and 2, respectively ($\beta = 5$). A detailed comparison of the modal second moments in both simulation and theory traced the discrepancy to the cross-correlation Γ_{k_0} for the zonal wavevector $\boldsymbol{k}_0 = (0, 1)$, which was four times larger in the simulation than in

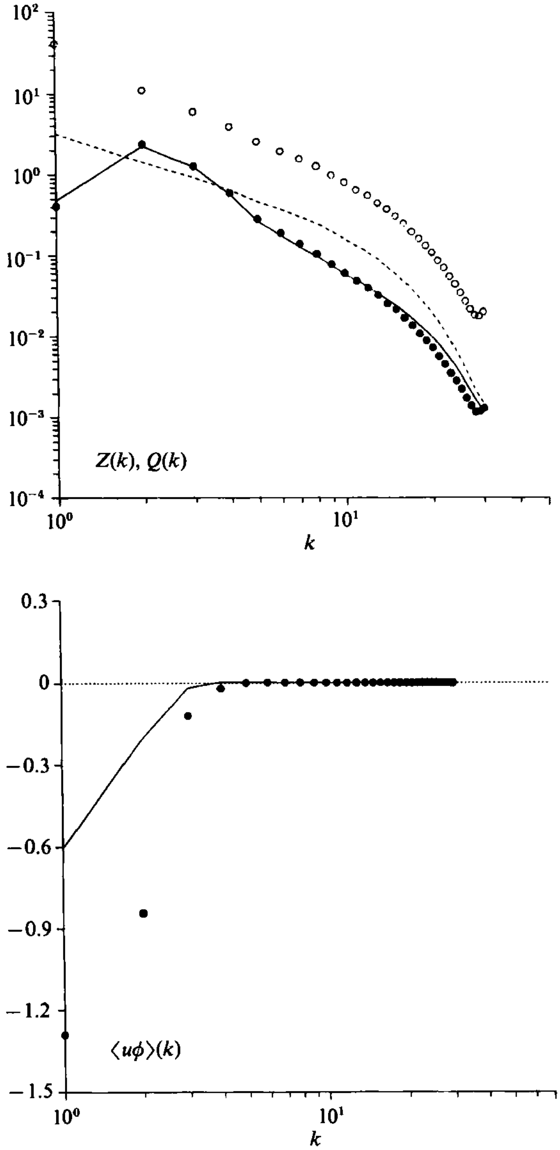

FIGURE 4. The vorticity, scalar variance and zonal flux spectra for $\beta = 5$, as in figure 2 but for $G = (2^{-\frac{1}{2}}, 2^{-\frac{1}{2}})$.

the closure. It was also noted that the theoretical Γ_k was a smooth function of $\boldsymbol{k}$ while the simulation showed a very abrupt peak at $\boldsymbol{k}_0$. In order to test whether the discrepancy was entirely due to this difference, the closure scalar variance equation (11) was evaluated using the closure Z_k and Γ_k, with the exception of the mode $\boldsymbol{k}_0$ for which the simulation Γ_{k_0} was substituted. The result was scalar variance agreement of the quality of §3.1, implying that, apart from Γ_{k_0}, the statistical theory was able to predict accurate second moments.

An examination of (12) reveals that Γ_{k_0} can be generated by the vorticity variance through the gradient term, or by nonlinear transfer into $\boldsymbol{k}_0$. Since reasonable agreement between closure and simulation Z_{k_0} was observed, the failure of the

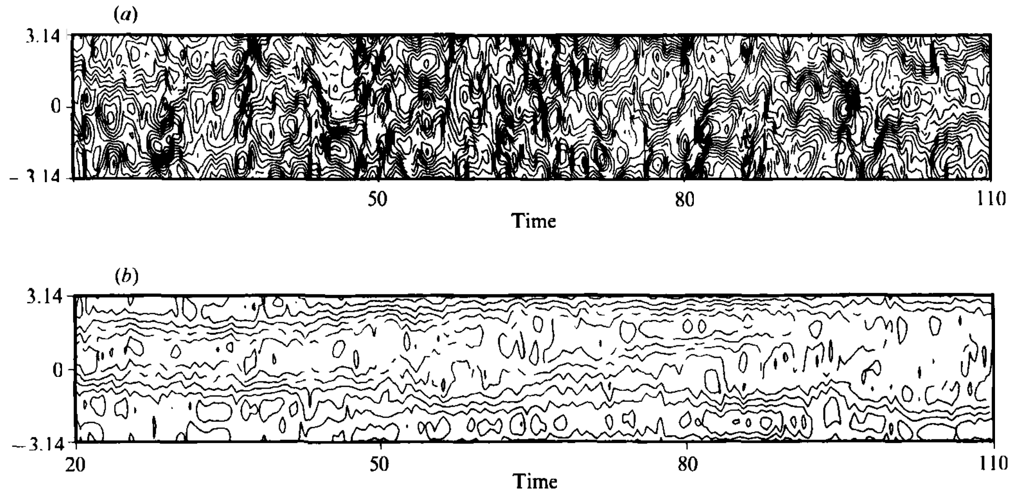

FIGURE 5. Time series of the zonal average velocity for (a) $\beta = 0$ and (b) $\beta = 5$.

statistical theory occurred in the transfer. It was apparently unable to reproduce the dynamics responsible for the abrupt peak in simulation Γ_k at $\boldsymbol{k}_0$. This mode is the most energetic of the zonal modes, which are distinguished by having zero Rossby frequency, implying constant phase in the linear case. In the nonlinear case, the closure assumes that both amplitude and phase vary on a timescale $\mu_{k_0}^{-1}$ characteristic of triple correlations. If, on the other hand, the phase becomes locked (i.e. fixed over a timescale long compared to $\mu_{k_0}^{-1}$), then the gradient source term in (6) conceivably produces a phase-locked response in ϕ_{k_0} which in turn produces large correlations and hence zonal fluxes.

To test this conjecture we have calculated time series of the zonal average of u for $\beta = 0$ and 5 which are displayed in figure 5. In terms of the large-scale eddy turnover time τ, the TFM decorrelation timescales $2\pi/\mu_{k_0}$ were 7τ ($\beta = 0$) and 15τ ($\beta = 5$). It is clear that the phase persistence of the zonal modes increases with β and is much longer than $2\pi/\mu_{k_0}$ when $\beta = 5$. This can be observed directly in the time series of the phase of $\boldsymbol{k}_0$ for $\beta = 0$ and 5 shown in figure 6. The same behaviour has also been noted at much lower resolution (down to $k_{\mathrm{T}} = 7$) for a variety of experiments with the meridional lengthscale being determined by the wavenumber of the energy-containing zonal modes. The existence of the jets was also seen to be rather sensitively dependent on the Ekman dissipation coefficient ν_0. At large ν_0 the jets were apparently dissipated on a shorter timescale than that required for their formation.

One perplexing aspect of this phenomenon is the quality of agreement between simulation and closure vorticity variances. Given the existence of important discrepancies in phase timescales, one might expect the closure enstrophy transfer to be less accurately represented. Although this apparently is not the case, the discrepancy makes its presence felt in the scalar equation where the effective forcing in the simulation is phase-locked, while the statistical effect of the assumed short-lived fluctuations in the closure is much weaker.

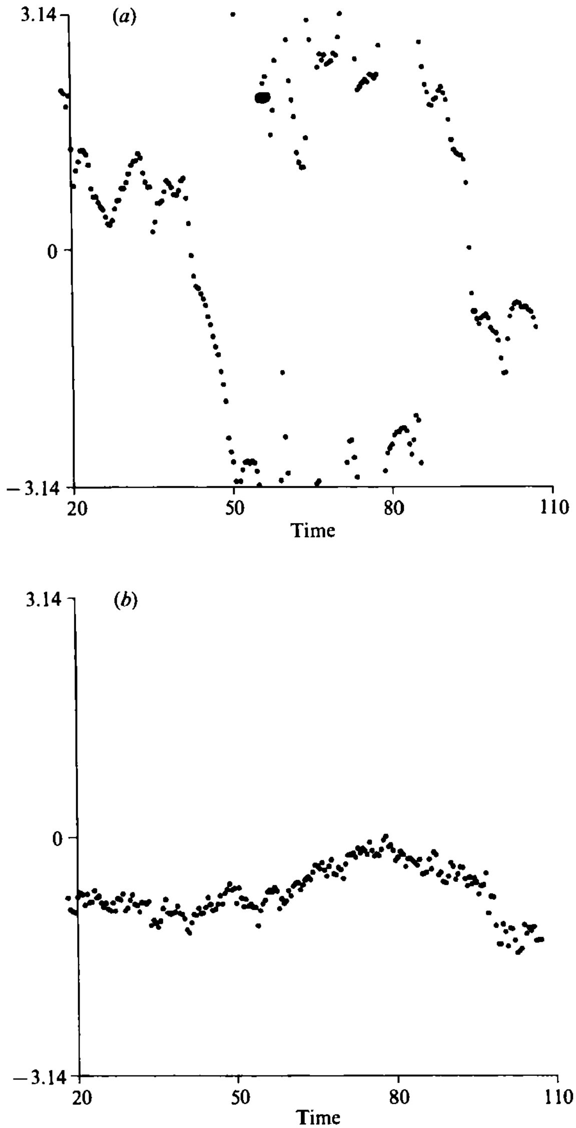

FIGURE 6. Time series of the phase of the mode $k_0 = (0,1)$ for (a) $\beta = 0$ and (b) $\beta = 5$.

4. Conclusions

The closure theory was able to reproduce approximately the vorticity variances of the simulation for the values of β considered here ($0 \leqslant \beta \leqslant 0.9$). With a meridional background scalar gradient the β-effect was found to induce vorticity–scalar correlations which were not well treated in the earlier theory of HK. After extending the theory to account for the case of strong β and to restore random Galilean invariance, the theoretical scalar, vorticity–scalar correlation and meridional diffusivity (K_{yy}) statistics were also in agreement with those of the simulation. The highly-simplified expression (2), motivated by the theory presented in HK, remains a good approximation when $C = 0.8$ and $B = 6$, and appears worthy of further investigation as a parameterization of K_{yy}.

In the presence of a significant zonal component to the scalar gradient, the agreement between simulation and theoretical scalar, correlation and zonal diffusivity (K_{xx}) statistics was rather poor. This failure of the statistical theory is due to the anomalous persistence of the zonal-mode phases in barotropic β-plane turbulence, i.e. zonal jets that wander only very slowly in the meridional direction. These persistence times must be accurately predicted by any theory that attempts to model zonal fluxes.

This research has been supported in part under the Office of Naval Research grant N00014-87-G-0262.

Appendix. Numerical details

A.1. *The evaluation of the closure equations*

Considering that the closure hypothesis assumes quasi-stationarity, and that it is more convenient in the simulations to perform time averaging over the stationary regime of one experiment than ensemble averaging, it was considered advantageous to restrict this study to statistically stationary states. The numerical approach adopted was to seek the stationary solution of the closure equations using the fewest possible number of iterations without necessarily describing transient behaviour accurately. Using the expression (9), μ_k is obtained from the vorticity variances Z_k by iteration. In order to describe accurately the time evolution of (10), (11) and (12), the μ_k must be iterated to stationarity at each timestep. We chose rather to use only one iteration per timestep, implying that our scheme does not approximate the early time behaviour. The steady state, however, remains unaffected.

The iterative scheme resembles forward-difference timestepping with the linear and nonlinear dissipation terms treated pseudo-analytically. For example, if

$$\frac{\partial}{\partial t} Z(t) = F(t) - \alpha(t)\, Z(t)$$

is the equation to be solved, the scheme treats $F(t)$ and $\alpha(t)$ as constants over one timestep then solves the equation analytically, i.e.

$$Z(t+\Delta t) = Z(t)\,\mathrm{e}^{-\alpha(t)\,\Delta t} + \frac{F(t)}{\alpha(t)}(1-\mathrm{e}^{-\alpha(t)\,\Delta t}). \tag{A 1}$$

The iterative procedure starts off with a rather poor guess for μ_k and a relatively small 'timestep'. After a few tens of iterations, the 'timestep' can be increased to infinity, implying an iteration of $Z^{n+1} = F(Z^n)/\alpha(Z^n)$. It was found to be particularly economical to carry out the procedure at low resolution (and large small-scale dissipation) and then to increase resolution (decrease dissipation) in steps until the desired result was achieved.

Although it is not, strictly speaking, part of the numerical detail, we would like to compare the implicit TFM formulation of μ_k with two other EDQNM explicit formulations. The first of these is an anisotropic generalization of the form used by Pouquet *et al.* (1975) in which μ_k is obtained from the shear due to scales larger than k^{-1}

$$\mu_k = \lambda_{\mathrm{P}}\Big(\sum_{|p|\leqslant|k|} Z_p\Big)^{\frac{1}{2}}. \tag{A 2}$$

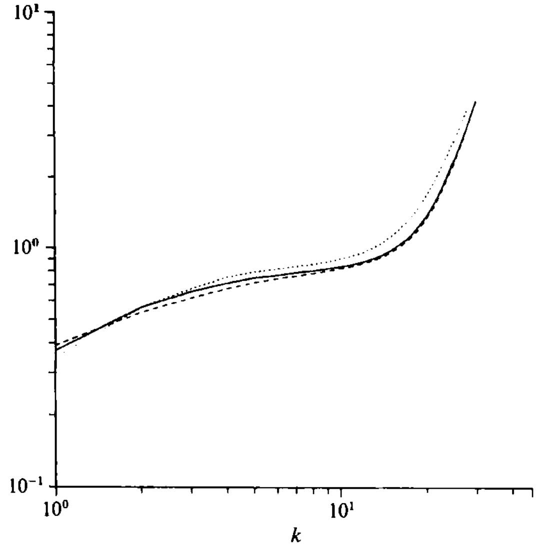

FIGURE 7. $\mu_k+\nu_k$ evaluated using the Test Field Model (——), equation (A 3) from Holloway (1987) (-----) and equation (A2) from Pouquet *et al.* (1975) (·····).

The other was proposed by Holloway (1987) as an approximation to the TFM which approaches (A 2) as $k \rightarrow \infty$, but which falls off more quickly with decreasing k in the range $0 \leqslant k \leqslant k_1$, where k_1^{-1} is the energy-containing scale. Setting the topographic terms, which that paper was mainly concerned with, to zero, and noting that his μ_k is our $\mu_k+\nu_k$, we obtain

$$\mu_k(\mu_k+\nu_k) = \lambda_{\mathrm{H}}^2 \sum_p \frac{k^2}{k^2+p^2} Z_p, \tag{A 3}$$

where it is clear that the positive solution is chosen. At large k both forms give approximately the same value provided $\lambda_{\mathrm{H}} = \lambda_{\mathrm{P}} = \lambda$. If these closures are to give the same Batchelor–Leith–Kraichnan constant in the enstrophy-cascading inertial range as the TFM, then $\lambda = 0.376g$ (Pouquet *et al.* 1975). The results of a comparison between (A 2) and (A 3), made with the vorticity variances from the TFM closure ($g = 1$, $\lambda = 0.376$, $\beta = 0$), show considerable agreement, suggesting that the closure is not extremely sensitive to the exact form of μ_k (see figure 7). From the numerical point of view it requires less computational effort to use an explicit formulation for non-stationary problems; however, it should be pointed out that, unlike the TFM, neither (A 2) nor (A 3) directly account for the influence of β.

A.2. *The direct numerical simulations*

The numerical simulations were performed using dealiased pseudo-spectral methods (Orszag 1971) as implemented in the model described in Ramsden, Whitfield & Holloway (1985). A leap-frog time scheme ($\Delta t = 0.005$) was employed on the nonlinear, forcing and gradient terms and the dissipation was applied through the use of exponential factors much as in (A 1). The isotropic forcing was of the form $F_k = f(k)\, g_k(t)$, where $f(k)$ was a constant in the waveband $2 \leqslant k \leqslant 4$ and zero elsewhere and $\langle g_k(t)\, g_k(s)\rangle = \delta(t-s)$. The effect of white-noise forcing on the leap-frog time scheme is to induce rapid decoupling. This was controlled with a numerical filter (Robert 1966) whose parameter was set to 0.1.

The spectra were calculated in the same manner in both the closure and simulations. If $S(k)$ is the power spectrum formed from the corresponding modal quantities S_k, then

$$S(k) = \frac{2\pi k}{N(k)} \sum_{k-\frac{1}{2} \leqslant |p| < k+\frac{1}{2}} S_p,$$

where $N(k)$ is the number of modes in the kth waveband, implying a smoothing out of the modal distribution over k-space. Since the closure solutions were very smooth functions of $\boldsymbol{k}$, this representation was considered preferable. Circular truncation was imposed at $k = 30.5$ in order to complete the outer waveband.

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